- The state ket for an arbitrary physical state.

$$|\alpha\rangle = \int_{-\infty}^{\infty} dx\, |x\rangle\langle x|\alpha\rangle$$

$\leftarrow$ continuum version of
$|\alpha\rangle = \sum_a |a\rangle\langle a|\alpha\rangle$

$\downarrow$

probability to find $|\alpha\rangle$ in the narrow interval around x.

$$= |\langle x|\alpha\rangle|^2 dx$$

probability density.

- In 3D, $|\vec{x}\rangle \equiv |x, y, z\rangle$

$$\tilde{x}|\vec{x}\rangle = x|\vec{x}\rangle, \quad \tilde{y}|\vec{x}\rangle = y|\vec{x}\rangle, \quad \tilde{z}|\vec{x}\rangle = z|\vec{x}\rangle$$

"simultaneous" eigenket!

$$\leftarrow [\tilde{x}_i, \tilde{x}_j] = 0.$$

(3) Translation "operator".

$$|\vec{x}\rangle \xrightarrow[\mathcal{J}(\delta\vec{x})]{} |\vec{x} + \delta\vec{x}\rangle$$

to make "infinitesimal" translation from $\vec{x}$ to $\vec{x} + \delta\vec{x}$

$$\mathcal{J}(\delta\vec{x})|\vec{x}\rangle = |\vec{x} + \delta\vec{x}\rangle$$

meaning: $\delta\vec{x}$ is too small to change anything else.

- effect of $\mathcal{J}(\delta\vec{x})$ on an arbitrary state ket $|\alpha\rangle$:

$$\mathcal{J}(\delta\vec{x})|\alpha\rangle = \mathcal{J}(\delta\vec{x})\int d^3x\, |\vec{x}\rangle\langle\vec{x}|\alpha\rangle$$

$$= \int d^3x\, |\vec{x}+\delta\vec{x}\rangle\langle\vec{x}|\alpha\rangle$$

express in terms of $|\vec{x}\rangle$

$$= \int d^3x\, |\vec{x}\rangle\langle\vec{x}-\delta\vec{x}|\alpha\rangle$$

|| shift the integration variable by $-\delta\vec{x}$.

effect of $\mathcal{J}(\delta\vec{x})$ on the expansion in terms of $|\vec{x}\rangle$

(integration is over "all" space)

kernel function $\langle\vec{x}|\alpha\rangle \longrightarrow$ shifted by $\boxed{-\delta\vec{x}}$.

(it's like the origin is shifted by $-\delta\vec{x}$ while the system is at $\vec{x}$)

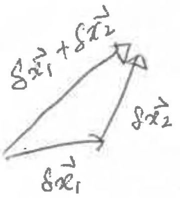
- properties of $\mathcal{J}(\delta\vec{x})$

① unitarity $\boxed{\mathcal{J}^{\dagger}(\delta\vec{x})\,\mathcal{J}(\delta\vec{x}) = 1}$

|| $\langle\alpha|\alpha\rangle = 1 = \langle\alpha|\mathcal{J}^{\dagger}(\delta\vec{x})\,\mathcal{J}(\delta\vec{x})|\alpha\rangle$

" the norm does not change! "

②



: $\boxed{\mathcal{J}(\delta\vec{x}_2)\,\mathcal{J}(\delta\vec{x}_1) = \mathcal{J}(\delta\vec{x}_1+\delta\vec{x}_2)}$

③ $\boxed{\mathcal{J}(-\delta\vec{x}) = \mathcal{J}^{-1}(\delta\vec{x})}$ || $\mathcal{J}(-\delta\vec{x})\,\mathcal{J}(\delta\vec{x}) = 1$

opposite-direction translation = inverse.

(of course.)

④ $\lim_{\delta\vec{x}\to 0} \mathcal{J}(\delta\vec{x}) = 1$ (No question!)

Since it's the infinitesimal translation.

We may write $\mathcal{J}$ as $\boxed{\mathcal{J}(\delta\vec{x}) \simeq 1 - i\,\underbrace{\mathbb{K}\cdot\delta\vec{x}}_{\tilde{K}_x\delta x + \tilde{K}_y\delta y + \tilde{K}_z\delta z}} + \cancel{O(\delta\vec{x}^2)}$

operator vector $\equiv (K_x, K_y, K_z)$

c-number vector

- Properties of $\mathbb{K}$ operator.

① unitarity of $\mathcal{J}$:

$$\mathcal{J}^\dagger(\delta\vec{x})\,\mathcal{J}(\delta\vec{x}) = (1 + i\mathbb{K}^\dagger\cdot\delta\vec{x})(1 - i\mathbb{K}\cdot\delta\vec{x})$$
$$\simeq 1 - i(\mathbb{K}-\mathbb{K}^\dagger)\cdot\delta\vec{x} + \cancel{O(\delta\vec{x}^2)}$$
$$= 1.$$

$\Rightarrow$ $\mathbb{K} = \mathbb{K}^\dagger$: $\boxed{\mathbb{K} \text{ is Hermitian}}$

② addition

$$\mathcal{J}(\delta\vec{x}_2)\,\mathcal{J}(\delta\vec{x}_1) = (1 - i\mathbb{K}\cdot\delta\vec{x}_2)(1 - i\mathbb{K}\cdot\delta\vec{x}_1)$$
$$\simeq 1 - i\mathbb{K}(\delta\vec{x}_1 + \delta\vec{x}_2)$$
$$= \mathcal{J}(\delta\vec{x}_1 + \delta\vec{x}_2) \quad : \underline{\text{OK}}.$$

☆☆☆

③ Important : relation between $\mathbb{K}$ and $(\tilde{x}, \tilde{y}, \tilde{z})$ operators.

commutation

notation

$\to (\tilde{x}, \tilde{y}, \tilde{z}) \equiv \tilde{x}_j \quad (j = 1, 2, 3)$

(i) $\tilde{x}_j\,\mathcal{J}(\delta\vec{x})|\vec{x}\rangle = \tilde{x}_j|\vec{x}+\delta\vec{x}\rangle = (x_j + \delta x_j)|\vec{x}+\delta\vec{x}\rangle$

(ii) $\mathcal{J}(\delta\vec{x})\,\tilde{x}_j|\vec{x}\rangle = x_j\,\mathcal{J}(\delta\vec{x})|\vec{x}\rangle = x_j|\vec{x}+\delta\vec{x}\rangle$

$$\Rightarrow \; [\tilde{x}_j, \, \mathcal{J}(\delta\vec{x})] \, |\vec{x}\rangle = \delta x_j \, |\vec{x} + \delta\vec{x}\rangle$$

$$\simeq \delta x_j \, |\vec{x}\rangle \quad \left(\text{up to the first order in } \delta\vec{x}\right)$$

Thus,

$$[\tilde{x}_j, \, \mathcal{J}(\delta\vec{x})] = \delta x_j \cdot \mathbb{1}$$

putting $\mathcal{J}(\delta\vec{x}) = 1 - i\mathbb{K}\cdot\delta\vec{x}$ into this eq. :

$$-i\,\tilde{x}_j\,(\tilde{K}_1\delta x_1 + \tilde{K}_2\delta x_2 + \tilde{K}_3\delta x_3)$$
$$+ i\,(\tilde{K}_1\delta x_1 + \tilde{K}_2\delta x_2 + \tilde{K}_3\delta x_3)\,\tilde{x}_j = \delta x_j \cdot \mathbb{1}$$

try $j=1$:

$$[-i\tilde{x}_1\tilde{K}_1 + i\tilde{K}_1\tilde{x}_1 - \mathbb{1}]\,\delta x_1$$
$$+ [-i\tilde{x}_1\tilde{K}_2 + i\tilde{K}_2\tilde{x}_1]\,\delta x_2$$
$$+ [-i\tilde{x}_1\tilde{K}_3 + i\tilde{K}_3\tilde{x}_1]\,\delta x_3 = 0$$

for arbitrary $\delta x_1, \delta x_2, \delta x_3$, this eq. should hold!

$$\Rightarrow \; [\tilde{x}_1, \tilde{K}_1] = i \;, \quad [\tilde{x}_1, \tilde{K}_{2,3}] = 0$$

try $j=2$, $j=3$, you will see.

$$\boxed{[\tilde{x}_i, \tilde{K}_j] = i\,\delta_{ij}}$$

($\mathbb{1}$ is omitted.)

Next question: What's "$\mathbb{K}$", then?

Ans. momentum.

(4) Momentum as a Generator of Translation

Put a name on operator "$\vec{K}$" !

It's like "momentum" in the "classical-quantum" correspondence.

$\mathbb{K} \propto \vec{\underset{\sim}{P}}$ "

* Canonical transformation in Classical Mech.

$$p_{\tilde{i}},\ q_{\tilde{i}} \quad \xrightarrow[H(q,p,t)]{\text{old} \quad \text{translation} \quad \text{new}} \quad H'(Q,P,t)$$

$\uparrow$ Hamiltonian

$$\left\{\begin{aligned} Q_{\tilde{i}} &\equiv Q_{\tilde{i}}(q,p,t) \\ &= q_{\tilde{i}} + \delta q_{\tilde{i}} \\ P_{\tilde{i}} &\equiv P_{\tilde{i}}(q,p,t) \\ &= p_{\tilde{i}} \end{aligned}\right.$$

This is what "Canonical" means.

To preserve the form of Hamilton's equation of motion,

$$\left[\begin{aligned} &\delta \int_{t_1}^{t_2} \left(p_{\tilde{i}}\dot{q}_{\tilde{i}} - H(q,p,t)\right) dt = 0 \quad : \text{Hamilton's principle.} \\ &\delta \int_{t_1}^{t_2} \left(P_{\tilde{i}}\dot{Q}_{\tilde{i}} - H'(Q,P,t)\right) dt = 0 \end{aligned}\right.$$

$$\Rightarrow \quad p_{\tilde{i}}\dot{q}_{\tilde{i}} - H = P_{\tilde{i}}\dot{Q}_{\tilde{i}} - H' + \frac{dF}{dt} \qquad \because\ \delta\left[F(t_2) - F(t_1)\right] = 0$$

: No variation at the end points.

$F(q_{\tilde{i}}, p_{\tilde{i}}, Q_{\tilde{i}}, P_{\tilde{i}}, t)$: a generating function of a canonical transformation.

only ~~three~~ two independent.

because $Q_{\tilde{i}} = Q_{\tilde{i}}(q,p,t)$, $P_{\tilde{i}} = P_{\tilde{i}}(q,p,t)$] two equations.

For the purpose of this particular translation

$$F = F_2(q,P,t) - Q_{\tilde{i}}P_{\tilde{i}} \qquad \left(\begin{aligned} Q_{\tilde{i}} &= q_{\tilde{i}} + \delta q_{\tilde{i}} \\ P_{\tilde{i}} &= p_{\tilde{i}} \end{aligned}\right.$$

: The generating function that we need!

for Legendre tr.

$$\frac{dF}{dt} = -P_i \dot{Q}_i - Q_i \dot{P}_i + \frac{dF_2}{dt}$$

time-independent problem.

$$= \frac{\partial F_2}{\partial q_i}\dot{q}_i + \frac{\partial F_2}{\partial P_i}\dot{P}_i + \cancel{\frac{\partial F_2}{\partial t}}$$

$$\Rightarrow \quad p_i \dot{q}_i - H = P_i \dot{Q}_i - H' + \frac{dF}{dt}$$

$$p_i \dot{q}_i - H = -Q_i \dot{P}_i - H' + \frac{\partial F_2}{\partial q_i}\dot{q}_i + \frac{\partial F_2}{\partial P_i}\dot{P}_i$$

$$\Rightarrow \quad \text{If } \frac{\partial F_2}{\partial q_i} = p_i \quad , \quad \frac{\partial F_2}{\partial P_i} = +Q_i$$

then , $H = H'$.

Now, try $F_2 = \vec{q}\cdot\vec{P} + \vec{P}\cdot\delta\vec{q}$

$$\Rightarrow \quad \frac{\partial F_2}{\partial q_i} = P_i = p_i$$

$$\frac{\partial F_2}{\partial P_i} = q_i + \delta q_i = Q_i$$

] OK!

$$\Rightarrow \quad F_2 = \vec{q}\cdot\vec{P} + \underset{\vec{p}}{\vec{P}}\cdot\delta\vec{q}$$

is the generating function that we're looking for!

The role of $\vec{q}\cdot\vec{P}$ term :

$\rightarrow$ $F_2 = \vec{q}\cdot\vec{P}$ gives $\begin{pmatrix} P_i = p_i \\ q_i = Q_i \end{pmatrix}$.

It's like $\boxed{\text{identity}}$ op.

$\therefore$ In Quantum counter part,

$$F_2 \xrightarrow[QM]{} I + \alpha \underset{\sim}{\vec{P}}\cdot\delta\vec{x}$$

$$\Longleftrightarrow \quad \mathcal{J}(d\vec{x}) = 1 - i\tilde{\mathbb{K}}\cdot\delta\vec{x}$$

$$\therefore \quad \alpha\vec{\underset{\sim}{p}} = -i\tilde{\mathbb{K}} \quad \rightarrow \quad \mathbb{K} = \frac{\vec{\underset{\sim}{p}}}{\text{(some const.)}}$$

some constant $\overset{?}{=}$ $\hbar$ (since $[\mathbb{K}] = [L]^{-1}$ and de Broglie's relation. $[\frac{2\pi}{\lambda} = \frac{p}{\hbar}] = [L]^{-1}$

$\Rightarrow$

$\boxed{\text{Define } \vec{\underset{\sim}{p}} \text{ operator (QM) such that} \Rightarrow \mathcal{J}(\delta\vec{x}) = 1 - i\vec{\underset{\sim}{p}}\cdot\delta\vec{x}/\hbar}$ ☆☆☆

Thus, $[\tilde{x}_i, \tilde{p}_j] = i\hbar\delta_{ij}$ (because we define $\vec{\underset{\sim}{p}}$ operator in that way.

$\Rightarrow$ uncertainty principle.

$$\langle\Delta\tilde{x}^2\rangle\langle\Delta\tilde{p}_x^2\rangle \geq \hbar^2/4.$$

Classical-Quantum correspondence, Again ...

$$[\ ,\ ]_{\text{classical}} \Rightarrow \frac{[\ ,\ ]_{\text{quantum}}}{i\hbar}$$

(Dirac)

—— Now we're ready to move on.

- Position translation operator with step Δx. (not infinitesimal)

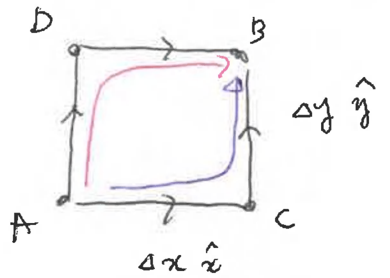
By factoring $\Delta x = N\,\delta x$ $\rightarrow$ infinitesimal.

$$\mathcal{J}(\Delta x\cdot\hat{x}) = \lim_{N\to\infty}\left(1 - i\frac{\tilde{p}_x}{\hbar}\cdot\left(\frac{\Delta x}{N}\right)\right)^N$$

$$= \exp\left[-\frac{i\tilde{p}_x\Delta x}{\hbar}\right]$$

Let's see the fundamental property of $\underset{\sim}{p}$ operator ...

- Successive translations in different directions



It does not matter whether it goes $A \to D \to B$ or $A \to C \to B$!

$$\Rightarrow \quad \mathcal{J}(\Delta y\,\hat{y})\,\mathcal{J}(\Delta x\,\hat{x}) = \mathcal{J}(\Delta x\,\hat{x} + \Delta y\,\hat{y})$$

$$\mathcal{J}(\Delta x\,\hat{x})\,\mathcal{J}(\Delta y\,\hat{y}) = \mathcal{J}(\Delta x\,\hat{x} + \Delta y\,\hat{y})$$

$$\Rightarrow \quad [\mathcal{J}(\Delta y\,\hat{y}), \mathcal{J}(\Delta x\,\hat{x})] = 0.$$

Since we know $\left[\begin{array}{l} \mathcal{J}(\Delta x\,\hat{x}) = \exp\left[-\dfrac{i\tilde{P}_x \Delta x}{\hbar}\right] \\ \mathcal{J}(\Delta y\,\hat{y}) = \exp\left[-\dfrac{i\tilde{P}_y \Delta y}{\hbar}\right] \end{array} \right]$

$$\Rightarrow \quad [\mathcal{J}(\Delta y\,\hat{y}), \mathcal{J}(\Delta x\,\hat{x})] = \left[1 - \frac{i\tilde{P}_y \Delta y}{\hbar} - \frac{1}{2!}\frac{\tilde{P}_y^2 \Delta y^2}{\hbar^2} + \cdots ,\; 1 - \frac{i\tilde{P}_x \Delta x}{\hbar} - \frac{1}{2!}\frac{\tilde{P}_x^2 \Delta x^2}{\hbar^2} + \cdots \right]$$

$$= -\frac{\Delta x \cdot \Delta y}{\hbar^2}[\tilde{P}_y, \tilde{P}_x] + \cdots$$

implication. $[\vec{P}, H] = 0$. $\Rightarrow$ H has momentum eigenstates!

$\therefore \quad [\tilde{P}_y, \tilde{P}_x] = 0$, in general $\boxed{[\tilde{P}_i, \tilde{P}_j] = 0}$

$\Rightarrow \quad \tilde{P}_x, \tilde{P}_y, \tilde{P}_z$ are mutually compatible. and thus has a simultaneous eigenket

$$|\vec{p}\rangle = |p_x, p_y, p_z\rangle \quad \Rightarrow \quad \begin{array}{l} \tilde{P}_x|\vec{p}\rangle = p_x|\vec{p}\rangle \\ \tilde{P}_y|\vec{p}\rangle = p_y|\vec{p}\rangle \\ \tilde{P}_z|\vec{p}\rangle = p_z|\vec{p}\rangle. \end{array}$$

also, one can show

$$[\vec{P}, \mathcal{J}(\delta\vec{x})] = 0 \quad \text{as well.}$$

(5) The Canonical Commutation Relations

$$[\tilde{x}_i, \tilde{x}_j] = 0 , \quad [\tilde{p}_i, \tilde{p}_j] = 0 , \quad [\tilde{x}_i, \tilde{p}_j] = i\hbar \delta_{ij}$$

other useful identities.

- $[A, A] = 0$, $[A, B] = -[B, A]$, $[A, c] = 0$ (c: c-number)
- $[A+B, C] = [A, C] + [B, C]$
- $[A, BC] = [A, B]C + B[A, C]$
- $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$

  (Jacobi identity).

1.7 Wave functions in position and momentum space.

(1) Position-Space wave function

→ Base kets = "position" kets : $\tilde{x}|x\rangle = x|x\rangle$

orthogonality : $\langle x|x'\rangle = \delta(x-x')$

completeness rel. $\int dx\, |x\rangle\langle x| = 1$

→ Wave function

a physical state $|\alpha\rangle = \int dx\, |x\rangle \langle x|\alpha\rangle$

$$= \int dx\, \psi_\alpha(x)\, |x\rangle .$$

≈ expansion coefficient of x-ket "localized" at x.

- wave function in position space

$$\boxed{\psi_\alpha(x) = \langle x|\alpha\rangle .}$$

probability for the particle to be found in $[x, x+dx]$ $= |\psi_\alpha|^2 dx$

- Inner product

$$\langle \beta|\alpha\rangle = \int dx\, \langle \beta|x\rangle\langle x|\alpha\rangle$$

$$= \int dx\, \psi_\beta^*(x)\, \psi_\alpha(x) .$$